Monotone Decrease of Characteristic Functions

Richard S. Ellis[1,2]

Let $F(x)$, $x = (x_1,..., x_n) \in \mathbb{R}^n$, $n \geqslant 1$, be an n-dimensional distribution function and define $f(t) = \int \exp(it \cdot x) \, dF(x)$. We prove the following result, which is used in the work of Monroe and Siegert.[1]

Theorem. We assume that

$$F(x) = F(-x), \qquad \text{all} \quad x \in \mathbb{R}^n \tag{1a}$$

$$\infty > \int (t \cdot x)^2 \, dF(x) > 0, \qquad \text{all} \quad t \in \mathbb{R}^n \tag{1b}$$

Then f is real and there exists $\delta > 0$ such that whenever $\epsilon > 0$, $t \in \mathbb{R}^n$, $0 < |t| < \delta/(1 + \epsilon)$, we have

$$f(t(1 + \epsilon)) < f(t) < 1 = f(0) \tag{2}$$

Thus in a suitable neighborhood of the origin, f is monotonically decreasing along rays through the origin.

Remark. Condition (1b) is equivalent to the nondegeneracy of F and the finiteness of the second moments $\int x_j^2 \, dF(x)$, $j = 1,..., n$.

Proof. We first prove (2) for $n = 1$ and then reduce the proof for general n to this case. Condition (1a) implies that f is real. Condition (1b) implies that $f \in C^2(\mathbb{R})$ and $f''(0) = -\int_R x^2 \, dF(x) < 0$ (prime denotes d/dt). By continuity, there exists $\delta > 0$ such that $f''(t) < 0$ for $|t| < \delta$. By (1a), $f'(0) =$

---

[1] Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts.
[2] On leave from Northwestern University, Evanston, Illinois.

$i \int_R x \, dF(x) = 0$. Thus for $0 < t < \delta, f'(t) = f'(t) - f'(0) = \int_0^t f''(s) \, ds < 0$. For $-\delta < t < 0, f'(t) = -\int_t^0 f''(s) \, ds > 0$. Now take $\epsilon > 0$ and t such that $0 < t < \delta/(1 + \epsilon)$. Then $f(t(1 + \epsilon)) - f(t) = \int_0^{\epsilon t} f'(t + s) \, ds < 0$. If $-\delta/(1 + \epsilon) < t < 0$, then $f(t(1 + \epsilon)) - f(t) = -\int_0^{-\epsilon t} f'(t - s) \, ds < 0$. To show $f(t) < f(0)$ is similar. This completes the proof for $n = 1$.

Now take $n \geqslant 2$. Again f is real. We write the argument t of f as $t = \alpha u$, where α is real and $u \in S^n$, the unit sphere in $\mathbb{R}^n$; thus, $f = f(\alpha, u)$. Condition (1b) implies that for each $u \in S^n$ (here prime denotes $d/d\alpha$)

$$f''(0, u) = -\int (u \cdot x)^2 \, dF(x) < 0$$

Hence by the joint continuity of $f''(\alpha, u)$ in α and u, there exists, for each $u \in S^n$, $\delta(u) > 0$ and a neighborhood $\mathcal{N}(u)$ of u so that $f''(\alpha, v) < 0$ for all $|\alpha| < \delta(u)$ and all $v \in \mathcal{N}(u)$. By the compactness of S^n, there exists an integer k and k points $u_1, \ldots, u_k$ so that

$$S^n = \bigcup_{i=1}^k \mathcal{N}(u_i)$$

We set $\delta = \min_{i=1,\ldots,k} \delta(u_i) > 0$; $f''(\alpha, u) < 0$ for all $|\alpha| < \delta$ and all $u \in S^n$. Condition (1a) implies that $f'(0, u) = 0$ for each $u \in S^n$. Thus we can carry over the proof for $n = 1$ and conclude that for $\epsilon > 0$

$$f(\alpha(1 + \epsilon), u) < f(\alpha, u), \qquad \text{all} \quad |\alpha| < \delta/(1 + \epsilon), \quad \text{all } u \in S^n$$

This completes the proof of the theorem.

ACKNOWLEDGMENT

REFERENCE

1. J. L. Monroe and A. J. F. Siegert, to be published.